Title: Finding Derivatives Using the Limit Definition
Class: Math 130 or Math 150
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Instructions to tutor: Read instructions and follow all steps for each problem exactly as given.
Keywords/Tags: Calculus, derivative, difference quotient, limit

Finding Derivatives Using the Limit Definition

Purpose:
This is intended to strengthen your ability to find derivatives using the limit definition.

Recall that an expression of the form $\dfrac{f(x)-f(a)}{x-a}$ or $\dfrac{f(x+h)-f(x)}{h}$ is called a **difference quotient**. For the definition of the derivative, we will focus mainly on the second of these two expressions. Before moving on to derivatives, let's get some practice working with the difference quotient.

The main difficulty is evaluating the expression $f(x+h)$, which seems to throw people off a bit.

Consider the function $f(x)=x^2-4x$. Let's evaluate this function at a few values.

$$f(2)=(2)^2-4(2)$$

$$f(0)=(0)^2-4(0)$$

$$f(-3)=(-3)^2-4(-3)$$

$$f(a)=(a)^2-4(a)$$

Note that we are just replacing the independent variable on each side of the equation with a particular value. So we should be able to do the same thing for $f(x+h)$: $f(x+h)=(x+h)^2-4(x+h)$

Now let's apply this to finding some difference quotients.

Example: Evaluate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the function $f(x)=x^2-4x$.

Now $\dfrac{f(x+h)-f(x)}{h}=\dfrac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}$.

Simplifying, $\dfrac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}=\dfrac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}=\dfrac{2xh+h^2-4h}{h}$.

Note that we can reduce this fraction to obtain $\dfrac{2xh+h^2-4h}{h}=\dfrac{h(2x+h-4)}{h}=2x+h-4$.

Example: Now it's your turn. Evaluate the difference quotient for $f(x) = x^2 + 3x - 5$.

First find $f(x+h) = (\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 5$

Now $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\left[(\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 5\right] - \left[(\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 5\right]}{h}$.

Did you get $\dfrac{[(x+h)^2 + 3(x+h) - 5] - [x^2 + 3x - 5]}{h}$? Good! Now simplify the numerator.

Did you get $\dfrac{2xh + h^2 + 3h}{h}$? If not, check that you distributed properly. Now factor the h out of the numerator and simplify the fraction.

You should have obtained $\dfrac{f(x+h) - f(x)}{h} = 2x + h + 3$. Try the next one on your own.

1. Evaluate the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for the function $f(x) = 2x^2 + 7x$.

Check your answer at the end of this document.

Now let's move on to finding derivatives. Recall the definition:

Limit-Definition of the Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Example: Find the derivative of $f(x) = x^2 - 4x$.

In a previous example, we found $\frac{f(x+h) - f(x)}{h} = 2x + h - 4$.

So $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}(2x + h - 4) = 2x - 4$.

Example: Find the derivative of $f(x) = x^3 + 5x^2 - 4$.

Now $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{[(x+h)^3 + 5(x+h)^2 - 4] - [x^3 + 5x^2 - 4]}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3xh^2 + 3x^2h + h^3) + (5x^2 + 10xh + 5h^2) - 4 - x^3 - 5x^2 + 4}{h}$$

$$= \lim_{h \to 0} \frac{3xh^2 + 3x^2h + h^3 + 10xh}{h}$$

$$= \lim_{h \to 0} \frac{h(3xh + 3x^2 + h^2 + 10x)}{h}$$

$$= \lim_{h \to 0}(3xh + 3x^2 + h^2 + 10x)$$

$$= 3x^2 + 10x$$

As you can see, the bulk of the work in finding the derivative is to evaluate and simplify the difference quotient.

You are on your own for the next two problems.

2. Find the derivative of each function using the limit definition.

   (a) $f(x) = x^2 + 3x - 5$ (Use your result from the first example on page 2 to help.)

   (b) $f(x) = 2x^2 + 7x$ (Use your result from the second example on page 2 to help.)

   (c) $f(x) = 4x^3 - 6x$ (Use the second example on page 3 as a guide.)

Check your answers at the end of this document.

Now you are ready to attempt these more challenging problems. You will need to employ the algebra skills you used in evaluating limits earlier, such as rationalizing techniques or adding rational expressions.

3. Find the derivative of each function using the limit definition.

(a) $f(x) = \sqrt{x}$

(b) $f(x) = \dfrac{1}{x}$

Check your answers – If you did not get these, consult a tutor for help.

1. $4x + 2h + 7$
2. (a) $f'(x) = 2x + 3$ (b) $f'(x) = 4x + 7$ (c) $f'(x) = 12x^2 - 6$
3. (a) $f'(x) = \dfrac{1}{2\sqrt{x}}$ (b) $f'(x) = -\dfrac{1}{x^2}$